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Nonlinear aspects of an internal gravity wave co-existing with an unstable mode associated with a Helmholtz velocity profile

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Recently Lindzen (1974) has proposed a model of a shear-layer instability which allows unstable modes to co-exist with radiating internal gravity waves. The model is an infinite, continuously stratified, Boussinesq fluid, with a simple jump discontinuity in the velocity profile. Linear stability theory shows that the model is stable for wavenumbers k such that $k^2 < N^2/2U^2$, where N is the Brunt–Väisälä frequency and $2U$ is the change in velocity across the discontinuity. For $N^2/2U^2 < k^2 < N^2/U^2$ an unstable mode may co-exist with an internal gravity wave. This paper examines the weakly nonlinear aspects of this model for wavenumbers k close to the critical wavenumber $N/2^{\frac{1}{2}}U$. An equation governing the evolution of the amplitude of the interfacial displacement is derived. It is shown that the interface may support a stable finite amplitude internal gravity wave.

1. Introduction

Clear-air turbulence is generally attributed to the Kelvin–Helmholtz instability of shear layers (q.v. Atlas *et al.* 1970; or review by Dutton & Panofsky 1970). Recently, however, Lindzen (1974) has drawn attention to the fact that some observations show the existence of internal gravity waves in the neighbourhood of shear layers (q.v. review by Ottersten, Hardy & Little 1973). Consequently Lindzen was led to consider a model of a shear layer which consisted of a simple jump discontinuity in velocity embedded in an infinite, continuously stratified, Boussinesq fluid. Such a model allows internal gravity waves to propagate away from the shear layer, and Lindzen has suggested that the energy flux associated with these waves may inhibit instability in the shear layer. Indeed, using linearized stability theory, Lindzen showed that, for a basic velocity discontinuity $2U$ and Brunt–Väisälä frequency N, perturbations with horizontal wavenumbers k are unstable when $k^2 > N^2/2U^2$. For $k^2 < N^2/U^2$, however, there also exists a neutral mode, or internal gravity wave. Thus Lindzen's model contains the interesting feature that an unstable mode may co-exist with an internal gravity wave. The implications of this for the energetics of the shear layer require a comprehensive discussion of nonlinear effects; conclusions based solely on the wave energy flux associated with the outwardly propagating waves

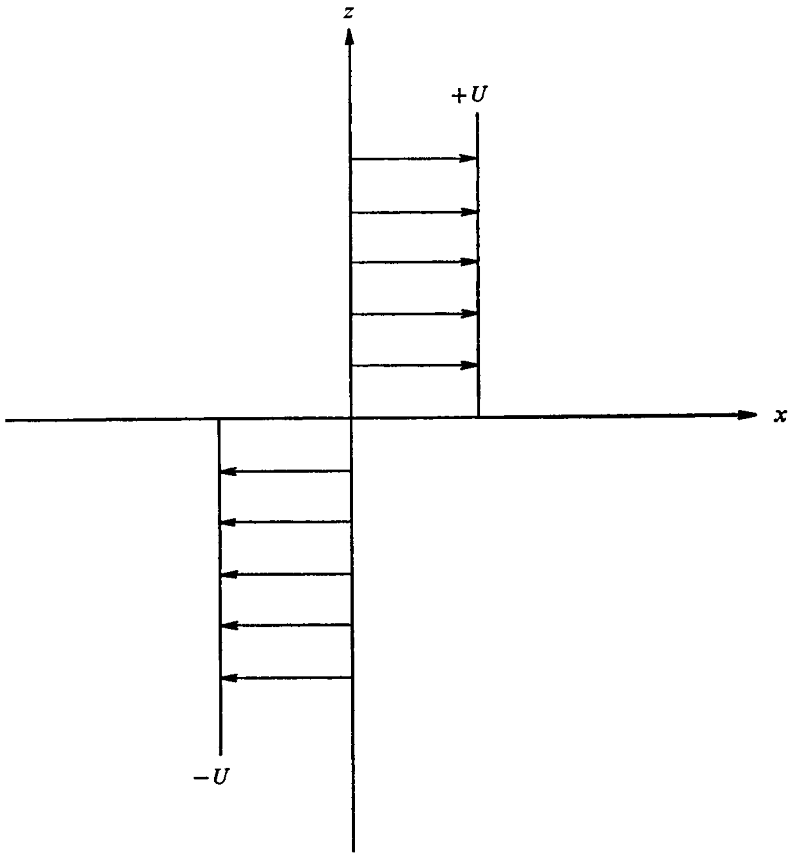

FIGURE 1. The basic velocity profile and co-ordinate system.

vis-à-vis the growth in energy of the unstable modes are likely to be erroneous (McIntyre & Weissman 1976). In the present paper we shall be concerned with just one aspect of the nonlinear effects. We propose to examine the weakly nonlinear regime associated with a single wavenumber k which is close to the critical wavenumber $N/2^{\frac{1}{2}}U$. Although modes with larger wavenumbers have faster growth rates, our hope is that the calculations presented here will throw some light on the nonlinear aspects of Lindzen's model. This hope is bolstered a little by the observations recorded in §4 (the same observations as were analysed by Lindzen), which show that the observed wavenumbers are close to the critical wavenumber. However, other observations (e.g. Metcalf & Atlas 1973) have indicated much larger wavenumbers, and it is well known that the Richardson number associated with the width of the shear layer is a crucial parameter for discussing clear-air turbulence. Lindzen's model contains no such Richardson number dependence. Nevertheless it is the simplest model possessing the feature of internal gravity waves co-existing with unstable modes, and for this reason, we propose to pursue its nonlinear aspects.

We shall assume that the basic state, in an infinite inviscid Boussinesq fluid, has a constant Brunt–Väisälä frequency N and a velocity, in the x direction, of

$\pm U$ in $z \gtrless 0$ (figure 1). It will be assumed that there is no variation in the y direction, as it may be shown that the stability criterion is independent of the wavenumber in the y direction. We shall use non-dimensional variables, based on a velocity scale U, a time scale N^{-1} and a length scale UN^{-1}; the reduced pressure (i.e. the deviation of the pressure from its hydrostatic value) is scaled by $\rho_1 U^2$, where ρ_1 is a reference density. Then the equations of motion are (e.g. Turner 1973, chap. 1)

$$u_x + w_z = 0, \tag{1.1}$$

$$\pm u_x + u_t + p_x = F_H = -uu_x - wu_z, \tag{1.2}$$

$$\pm w_x + w_t + p_z + r = F_V = -uw_x - ww_z, \tag{1.3}$$

$$\pm r_x + r_t - w = G = -ur_x - wr_z. \tag{1.4}$$

Here u and w are the x and z components of the perturbed velocity, p is the reduced pressure and r is the buoyancy (i.e. $g(\rho - \rho_0)/\rho_0$ scaled by UN, where $\rho_0(z)$ is the density in the basic state). The equations have been written in a form in which the linear terms are on the left-hand side and the nonlinear terms, represented by F_H, F_V and G, are on the right-hand side. The symbols $\pm$ indicate the regions $z \gtrless \zeta$, where $z = \zeta$ is the equation of the perturbed interface. If the variables on the left-hand side are eliminated in favour of w, it follows that

$$L^{\pm} w = M^{\pm}, \tag{1.5}$$

where $L^{\pm}$ are the *linear* operators

$$L^{\pm}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = -\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{\partial^2}{\partial x^2} \tag{1.6}$$

and $M^{\pm}$ are the nonlinear expressions

$$M^{\pm} = \frac{\partial^2}{\partial x^2}\left(G - \frac{\partial F_V}{\partial t} \mp \frac{\partial F_V}{\partial x}\right) + \frac{\partial^2}{\partial x\, \partial z}\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right) F_H. \tag{1.7}$$

The boundary conditions at the interface $z = \zeta$ are continuity of the pressure, and the kinematic condition

$$\zeta_t \pm \zeta_x + u\zeta_x - w = 0 \quad \text{at} \quad z = \zeta. \tag{1.8}$$

We anticipate that ζ will be small, and expand these conditions in a Taylor series about $z = 0$. Thus (1.8) becomes

$$\zeta_t \pm \zeta_x - w = H^{\pm} \quad \text{at} \quad z = 0\pm, \tag{1.9}$$

where $H^{\pm}$ are the nonlinear expressions

$$H^{\pm} = \{\zeta w_z + \tfrac{1}{2}\zeta^2 w_{zz} + \tfrac{1}{6}\zeta^3 w_{zzz} + \ldots - \zeta_x u - \zeta\zeta_x u_z - \tfrac{1}{2}\zeta^2 \zeta_x u_{zz} - \ldots\} \quad \text{at} \quad z = 0\pm. \tag{1.10}$$

Using (1.1), we may write $H^{\pm}$ in the alternative form

$$H^{\pm} = \{-(u\zeta)_x - (\tfrac{1}{2}\zeta^2 u_z)_x - (\tfrac{1}{6}\zeta^3 u_{zz})_x - \ldots\} \quad \text{at} \quad z = 0\pm. \tag{1.11}$$

The pressure condition becomes

$$[p]_{-}^{+} = Q, \tag{1.12}$$

where Q is the nonlinear expression

$$Q = -\zeta[p_z]^{+}_{-} - \tfrac{1}{2}\zeta^2[p_{zz}]^{+}_{-} - \tfrac{1}{6}\zeta^3[p_{zzz}]^{+}_{-} - \dots \tag{1.13}$$

Here $[p]^{+}_{-}$ etc. denote the discontinuities in p etc. across $z = 0$.

The linearized equations, discussed by Lindzen (1974), are now obtained by formally putting F_H, F_V, G, $H^\pm$ and Q equal to zero. Seeking solutions proportional to $\exp\{ik(x-ct)\}$, we find that

$$w = \alpha A^\pm \exp\{ik(x-ct) \pm in^\pm z\} \quad \text{in} \quad z \gtrless 0, \tag{1.14}$$

$$\zeta = \alpha A \exp\{ik(x-ct)\}. \tag{1.15}$$

Here $A^\pm$ and A are constant amplitudes, while α is a small parameter introduced as an appropriate measure of the magnitude of ζ. We shall assume throughout that k is positive. The constants $n^\pm$ are given by

$$(n^\pm)^2 = (c \mp 1)^{-2} - k^2. \tag{1.16}$$

The appropriate branch of $n^\pm$ is selected by applying a radiation condition. In linearized problems it is customary to obtain a radiation condition by requiring the wave energy flux, or the group velocity, to be outward. However, in the nonlinear context of subsequent sections, conditions at infinity cannot be obtained by considerations of wave energy flux alone. Instead, we shall require that our solutions decay exponentially when c_i (the imaginary part of c) takes small positive values. This condition is motivated by considering an appropriate initial-value problem. Lighthill (1960) has shown that in linearized problems this condition is equivalent to conditions based on group-velocity criteria. Let

$$n^\pm = n_r^\pm + in_i^\pm, \quad c = c_r + ic_i. \tag{1.17}$$

Then, our radiation condition for the solution (1.14) is either

$$n_i^\pm > 0 \tag{1.18}$$

or

$$n_i^\pm = 0, \quad n_r^\pm(c_r \mp 1) < 0. \tag{1.19}$$

Next, the linearized boundary conditions show that

$$A^\pm = -ik(c \mp 1) A \tag{1.20}$$

and that

$$n^+(c-1)^2 + n^-(c+1)^2 = 0. \tag{1.21}$$

This is the dispersion relation which determines c as a function of k. The solutions are

$$c = 0 \quad \text{for} \quad 0 < k^2 \leqslant 1 \tag{1.22}$$

and

$$c^2 = (2k^2)^{-1} - 1 \quad \text{for} \quad k^2 > \tfrac{1}{4}. \tag{1.23}$$

The solution (1.22) represents an internal gravity wave (stationary in the present frame of reference); the restriction on k is obtained from the radiation condition (1.19). (The vertical group velocity of this wave has magnitude $(k^2 - k^4)^{\frac{1}{2}}$, and is directed away from the interface in both media.) As it consists only of waves propagating away from the interface, it may be regarded as a special case of over-reflexion (Acheson 1976). The solution (1.23) is also an internal gravity wave for $\frac{1}{4} < k^2 < \frac{1}{2}$; the lower bound on k is obtained from the radiation condition (1.19), and implies that the phase speed c is bounded by unity. This mode was not discussed by Lindzen, who put c_r equal to zero. For $k^2 > \frac{1}{2}$, the solution (1.23) represents an unstable mode for which c_r is zero and c_i increases from zero to unity as k^2 increases from $\frac{1}{2}$ to infinity. The critical wavenumber k_c which separates unstable modes from stable modes is given by

$$k_c^2 = \tfrac{1}{2}. \tag{1.24}$$

The interesting feature of these solutions is the co-existence of internal gravity waves with unstable modes when k lies between k_c and unity.

In the nonlinear analysis of subsequent sections, we shall consider wavenumbers k close to the critical wavenumber k_c. It is apparent from (1.23) that c_i^2 is approximately $2(k-k_c)/k_c$. We anticipate that c_i is $O(\alpha)$ and hence define

$$k = k_c(1+\alpha^2\gamma), \tag{1.25}$$

where γ is $O(1)$ with respect to the amplitude parameter α. We shall attempt to describe the nonlinear effects by allowing the amplitudes to evolve slowly in time, on a time scale $O(\alpha^{-1})$. Thus we shall introduce the slow time variable

$$T = \alpha t, \tag{1.26}$$

and allow the amplitude A to depend on T. This is a familiar feature of weakly nonlinear stability calculations, and this technique has been applied to classical Kelvin–Helmholtz problems by Drazin (1970). Away from the interface, this slow time modulation will cause a slow modulation in space, and so we shall introduce

$$Z = \alpha z, \tag{1.27}$$

and allow $A^{\pm}$ to depend on T and Z. We note that c is zero when $k = k_c$, and that

$$n^{\pm} = \pm k_c \quad \text{when} \quad k = k_c. \tag{1.28}$$

In §§2 and 3 we shall describe the weakly nonlinear analysis, and in §4 we shall discuss the results of this analysis as it affects the evolution of the amplitude A. For reasons which we shall discuss in §3, the analysis will be carried through to $O(\alpha^4)$.

2. Weakly nonlinear theory

Motivated by the discussion at the end of the last section we are led to consider solutions of the form

$$\zeta = \sum_{m=-\infty}^{m=\infty} \zeta_m(T)\exp\{imkx\} + \text{c.c.}, \tag{2.1}$$

$$w = \sum_{m=-\infty}^{m=\infty} w_m(T,z,Z)\ \exp\{imkx\} + \text{c.c.} \tag{2.2}$$

Here $\zeta_m = \bar{\zeta}_{-m}$ etc. These expressions, and the corresponding expressions for u, r and p, are then substituted into the equations (1.5) and the boundary conditions (1.9) and (1.12). The result is, on equating like Fourier components,

$$L^{\pm}\left(\alpha\frac{\partial}{\partial T}, imk, \frac{\partial}{\partial z}+\alpha\frac{\partial}{\partial Z}\right) w_m = M^{\pm}_m, \quad z \gtrless 0, \tag{2.3}$$

$$\alpha\frac{\partial \zeta_m}{\partial T} \pm imk\zeta_m - w_m = H^{\pm}_m, \quad z = 0\pm, \tag{2.4}$$

$$[p_m]^{+}_{-} = Q_m, \quad z = 0. \tag{2.5}$$

Here the operators $L^{\pm}$ are defined by (1.6), and $M^{\pm}_m$, $H^{\pm}_m$ and Q_m are the mth Fourier components of the nonlinear terms $M^{\pm}$, $H^{\pm}$ and Q defined in (1.7), (1.10) and (1.13) respectively. In these equations k is expanded about k_c, according to (1.25). Throughout the subsequent analysis the superscript $\pm$ indicates an expression defined in $z \gtrless 0$.

For the Fourier component $m = 1$, it may be shown that $M^{\pm}_1$ are $O(\alpha^3)$, a result which we shall verify *a posteriori*. Hence

$$L^{\pm}\left(\alpha\frac{\partial}{\partial T}, ik_c, \frac{\partial}{\partial z}+\alpha\frac{\partial}{\partial Z}\right) w_1 = O(\alpha^3). \tag{2.6}$$

The appropriate solution for w_1 is thus

$$w_1 = \alpha A^{\pm}(T, Z)\exp\{ik_c z\} + O(\alpha^3), \tag{2.7}$$

where

$$\partial A^{\pm}/\partial Z = \mp 2\partial A^{\pm}/\partial T + O(\alpha). \tag{2.8}$$

The solution (2.7) should be compared with the unmodulated linearized solution (1.14), for which $n^{\pm}$ are just $\pm k_c$ by (1.28). The result (2.8) follows most readily by substituting (2.7) into (2.6), and examining the term $O(\alpha^2)$. Its significance is that it demonstrates that, to leading order in α, modulations in the amplitude $A^{\pm}$ propagate vertically upwards (downwards) into $Z > 0$ (< 0) at the group velocity corresponding to the wavenumber $k = k_c$. Indeed it is well known that the vertical group velocity for an internal gravity wave of vertical wavenumber n, horizontal wavenumber k and intrinsic frequency $\omega \mp k$ is (Phillips 1966, p. 175)

$$-n(\omega \mp k)/(n^2+k^2). \tag{2.9}$$

Here ω equals kc and is zero, while k is k_c and n is also k_c; hence the group velocity from (2.9) is $\pm\frac{1}{2}$. This discussion demonstrates that our solution (2.7) may be regarded as an internal gravity wave propagating vertically upwards (downwards) into $Z > 0$ (< 0). This is a consequence of our expansion being centred around the critical wavenumber k_c, and it is certainly not true that unstable modes for which k differs greatly from k_c can be regarded as radiating waves. For a detailed discussion of this point see McIntyre & Weissman (1976).

Turning next to the boundary condition (2.4) for $m = 1$, it may be shown that $H^{\pm}_1$ are $O(\alpha^3)$, a result which we shall verify *a posteriori*. Substituting (2.7) into (2.4) and relabelling

$$\zeta_1(T) = \alpha A(T), \tag{2.10}$$

it follows that

$$A^{\pm}(T, 0) = \pm ikA + \alpha\,\partial A/\partial T + O(\alpha^2). \tag{2.11}$$

Once the nonlinear terms $M_1^\pm$, $H_1^\pm$ and Q_1 have been evaluated, the boundary condition (2.5) leads to an amplitude equation for $A(T)$. However, the calculations leading to this equation will be deferred to §3, and the remainder of this section will be concerned with the Fourier components $m = 0$ and 2, which we anticipate to be at least $O(\alpha^2)$. The remaining Fourier components ($m \geq 3$) are at least $O(\alpha^3)$, and it may be shown that they do not contribute in the weakly nonlinear situation being considered here.

We turn now to the Fourier component $m = 2$, and putting $m = 2$ in (2.3), we have

$$L^\pm\left(\alpha\frac{\partial}{\partial T}, 2ik, \frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial Z}\right) w_2 = M_2^\pm, \quad z \gtrless 0. \tag{2.12}$$

For subsequent purposes it will be sufficient to know w_2 to within an error $O(\alpha^4)$, and hence we need only consider the contribution of w_1, u_1, r_1 and p_1 to F_{H2}, F_{V2} and G_2 [and hence to $M_2^\pm$ by (1.7)]. Now (1.1) implies that

$$iku_1 + \partial w_1/\partial z + \alpha\,\partial w_1/\partial Z = 0. \tag{2.13}$$

Also, it may be shown, and will be verified *a posteriori*, that F_{H1}, F_{V1} and G_1 are all $O(\alpha^3)$. Hence, in particular, (1.4) implies that

$$w_1 = \pm ikr_1 + \alpha\,\partial r_1/\partial T = O(\alpha^3). \tag{2.14}$$

Using the results (2.13) and (2.14) it may easily be shown that F_{H2}, F_{V2} and G_2 are each $O(\alpha^4)$, and hence $M_2^\pm$ are $O(\alpha^4)$. Thus, to leading order, the equation for w_2 is

$$L^\pm(0, 2ik_c, \partial/\partial z)\, w_2 \equiv \partial^2 w_2/\partial z^2 - w_2 = O(\alpha)\, w_2 + O(\alpha^4). \tag{2.15}$$

Here, the first term on the right-hand side is a term representing various operations (such as $\alpha\,\partial/\partial T$) applied to w_2 which are at least $O(\alpha)$ compared with w_2, and the second term is $M_2^\pm$. Thus the appropriate solution for w_2, which is bounded as $z \to \pm\infty$, is

$$w_2 = \alpha^3 A_2^\pm(T, Z) \exp(\mp z) + O(\alpha^4). \tag{2.16}$$

Here we have inserted a factor α^3 in anticipation of the fact that the boundary conditions will show that w_2 is $O(\alpha^3)$ (rather than $O(\alpha^2)$ as might have been expected). Also, it follows from (1.1), (1.2) and (1.4) that

$$\left.\begin{aligned} u_2 &= \mp\alpha^3 ik_c A_2^\pm \exp(\mp z) + O(\alpha^4),\\ p_2 &= \alpha^3 ik_c A_2^\pm \exp(\mp z) + O(\alpha^4),\\ r_2 &= \mp\alpha^3 ik_c A_2^\pm \exp(\mp z) + O(\alpha^4).\end{aligned}\right\} \tag{2.17}$$

We see therefore that the second Fourier components decay exponentially away from the interface, and unlike the first Fourier component (2.7), are not capable of transporting energy away from the interface.

Next we must consider the boundary conditions for $m = 2$. First we relabel

$$\zeta_2 = \alpha^2 A_2(T). \tag{2.18}$$

Then, from (2.4) and (2.5), using (2.16) and (2.17), we have

$$\alpha^2\{\alpha\, \partial A_2/\partial T \pm 2ikA_2 - \alpha A_2^{\pm}(T,0)\} + O(\alpha^4) = H_2^{\pm}, \tag{2.19}$$

$$\alpha^3 ik_c\{A_2^{+}(T,0) - A_2^{-}(T,0)\} = Q_2. \tag{2.20}$$

To calculate $H_2^{\pm}$, we first observe that (1.11) implies that

$$H_2^{\pm} = -2iku_1\zeta_1 + O(\alpha^4). \tag{2.21}$$

Then, from (2.7), (2.8), (2.11) and (2.14), it follows that

$$H_2^{\pm} = \mp\alpha^2 A^2 - \alpha^3 2ik_c A\, \partial A/\partial T + O(\alpha^4). \tag{2.22}$$

Similarly it may be shown that

$$Q_2 = \alpha^3 4ik_c A\, \partial A/\partial T + O(\alpha^4). \tag{2.23}$$

Substituting (2.22) into (2.19), and (2.23) into (2.20), it follows that

$$A_2 = ik_c A^2 - \alpha 2ik_c A\, \partial A/\partial T + O(\alpha^2), \tag{2.24}$$

$$A_2^{\pm}(T,0) = (4ik_c \pm 2)\, A\, \partial A/\partial T + O(\alpha). \tag{2.25}$$

We have now confirmed that $A_2^{\pm}$ are $O(1)$, and so w_2, u_2, p_2 and r_2 are all $O(\alpha^3)$, as we anticipated earlier, although the interfacial displacement ζ_2 is $O(\alpha^2)$. Equation (2.25) determines $A_2^{\pm}(T,0)$ in terms of $A(T)$, but the behaviour of $A_2^{\pm}(T,Z)$ with respect to the co-ordinate Z is undetermined at this stage. The appropriate equation to determine this behaviour may be obtained by examining the $O(\alpha^4)$ terms in (2.16). However, we shall not display this calculation here as it transpires that a knowledge of $A_2^{\pm}(T,0)$ alone is sufficient when considering the amplitude equation for $A(T)$.

The Fourier component $m = 0$, or the mean flow, may be obtained by putting $m = 0$ in (2.3)–(2.5). However it is more instructive to observe that the equations governing the mean flow may also be obtained by averaging, over one wavelength, with respect to x. Thus the Fourier component $f_0(T,z,Z)$ of some field variable $f(T,x,z,Z)$ may be defined by

$$f_0 = \langle f\rangle = \frac{k}{2\pi}\int_0^{2\pi/k} f\,dx. \tag{2.26}$$

Applying this averaging operation to (1.1), it follows immediately that

$$w_0 = 0. \tag{2.27}$$

Similarly the result of averaging (1.2)–(1.4) is

$$\alpha\frac{\partial u_0}{\partial T} = F_{H0} = -\left(\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial Z}\right)\langle uw\rangle, \tag{2.28}$$

$$\frac{\partial p_0}{\partial z} + \alpha\frac{\partial p_0}{\partial Z} + r_0 = F_{V0} = -\left(\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial Z}\right)\langle w^2\rangle, \tag{2.29}$$

$$\alpha\frac{\partial r_0}{\partial T} = G_0 = -\left(\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial Z}\right)\langle rw\rangle. \tag{2.30}$$

Turning next to the boundary conditions, we see immediately from (1.11) that $H_0^{\pm}$ are identically zero, and hence, on averaging (1.9), it follows that

$$\zeta_0 = 0. \tag{2.31}$$

Finally, averaging (1.12) shows that

$$[p_0]_-^+ = Q_0. \tag{2.32}$$

To calculate the nonlinear terms F_{H0}, F_{V0}, G_0 and Q_0 to within the required error, we need only consider the contribution from ζ_1, w_1, u_1, r_1 and p_1. We find that

$$F_{H0} = 2\alpha^3 \frac{\partial}{\partial Z}\left\{|A^{\pm}|^2 - \alpha ik\left(\bar{A}^{\pm}\frac{\partial A^{\pm}}{\partial Z} - A^{\pm}\frac{\partial \bar{A}^{\pm}}{\partial Z}\right)\right\} + O(\alpha^5), \tag{2.33}$$

$$F_{V0} = -2\alpha^3 \partial|A^{\pm}|^2/\partial Z + O(\alpha^5), \tag{2.34}$$

$$G_0 = 2\alpha^4 \frac{\partial^2}{\partial Z\,\partial T}\left\{-|A^{\pm}|^2 \pm 2\alpha ik\left(A^{\pm}\frac{\partial \bar{A}^{\pm}}{\partial T} - \bar{A}^{\pm}\frac{\partial A^{\pm}}{\partial T}\right)\right\} + O(\alpha^6), \tag{2.35}$$

$$Q_0 = 4\alpha^3 ik\left(A\frac{\partial \bar{A}}{\partial T} - \bar{A}\frac{\partial A}{\partial T}\right) + O(\alpha^4). \tag{2.36}$$

It is now apparent that (2.28) determines u_0, while (2.30) determines r_0; then (2.29) plus the boundary condition (2.32) determines p_0. Also, since (2.33)–(2.35) show that F_{H0}, F_{V0} and G_0 are independent of z, it follows that u_0, p_0 and r_0 are independent of z, at least to within the required error. We find that

$$r_0 = 2\alpha^3 \frac{\partial}{\partial Z}\left\{-|A^{\pm}|^2 \pm 2\alpha ik\left(A^{\pm}\frac{\partial \bar{A}^{\pm}}{\partial T} - \bar{A}^{\pm}\frac{\partial A^{\pm}}{\partial T}\right)\right\} + O(\alpha^5), \tag{2.37}$$

$$p_0 = \mp 4\alpha^3 ik\left(A^{\pm}\frac{\partial \bar{A}^{\pm}}{\partial T} - \bar{A}^{\pm}\frac{\partial A^{\pm}}{\partial T}\right) + O(\alpha^4). \tag{2.38}$$

It may easily be verified, using (2.11), that p_0, as given by (2.38), will satisfy the boundary condition (2.32). To find u_0, we use (2.8), and hence

$$u_0 = \mp 4\alpha^2|A^{\pm}|^2 + O(\alpha^3). \tag{2.39}$$

The 'constant' of integration (here an arbitrary function of Z) has been set equal to zero, as we are assuming that the disturbance originates at the interface at time $T = 0$. In the next section we shall need to calculate the $O(\alpha^3)$ term in (2.39) explicitly, but we cannot do this until the $O(\alpha)$ term in (2.8) is known explicitly. This calculation will be displayed in the next section.

Our result for the mean flow shows that p_0 and r_0 are $O(\alpha^3)$, while the mean velocity u_0 is $O(\alpha^2)$. Further, it is apparent from (2.28) that the acceleration of the mean velocity is simply due to the gradient of the Reynolds-stress component $\langle uw \rangle$, where the input for $\langle uw \rangle$ is just the internal gravity wave (2.7), which is propagating vertically upwards (downwards) into $Z > 0$ (< 0) at the group velocity $\pm\frac{1}{2}$ [see (2.8)]. For a general analysis of the equations governing the mean flow associated with a propagating internal gravity wave, see Grimshaw (1974). In the present context, it may be shown that the solution (2.39) is just that needed to ensure that the *total* energy flux in the vertical direction is zero, although there is a non-zero *wave* energy flux in the vertical direction associated with the internal gravity wave (2.7) (see Acheson (1976) for a detailed discussion of this aspect in a more general context than that considered here). The implications of (2.39) for the energetics of this system have been discussed recently by McIntyre & Weissman (1976).

3. Derivation of the amplitude equation

In this section we shall derive the amplitude equation for $A(T)$ which is obtained by considering the Fourier component $m = 1$. First, let us consider the equation (2.3) for w_1, and calculate the nonlinear terms $M_1^{\pm}$ to within an error $O(\alpha^5)$. We let

$$F_{H1} = F_{H1}^{(0)} + F_{H1}^{(2)} + O(\alpha^5), \text{ etc.,} \tag{3.1}$$

where a superscript (0) indicates the contribution to F_{H1}, etc., from the interaction of the Fourier components $m = 0$ and $m = 1$, and a superscript (2) indicates the contribution from the interaction of the Fourier components $m = 2$ and $m = 1$; the higher Fourier components will contribute only to the error term. Since w_2, u_2 and r_2 are $O(\alpha^3)$, $F_{H1}^{(2)}$ etc. are $O(\alpha^4)$, and are given by

$$\left.\begin{aligned} F_{H1}^{(2)} &= -iku_2\bar{u}_1 - w_2 D\bar{u}_1 - \bar{w}_1 Du_2, \\ F_{V1}^{(2)} &= iku_2\bar{w}_1 - 2ikw_2\bar{u}_1 - w_2 D\bar{w}_1 - \bar{w}_1 Dw_2, \\ G_1^{(2)} &= iku_2\bar{r}_1 - 2ik\bar{u}_1 r_2 - w_2 D\bar{r}_1 - \bar{w}_1 Dr_2. \end{aligned}\right\} \tag{3.2}$$

Here D denotes the *total* derivative with respect to z:

$$D = \partial/\partial z + \alpha\,\partial/\partial Z. \tag{3.3}$$

Then, using (2.13), (2.14), (2.16) and (2.17), we can show that

$$M_1^{\pm(2)} = O(\alpha^5). \tag{3.4}$$

Thus, remarkably, the second Fourier components do not contribute to $M_1^{\pm}$, at least not to $O(\alpha^4)$.

Next, since u_0 is $O(\alpha^2)$ but r_0 is $O(\alpha^3)$ and w_0 is zero, we see that

$$\left.\begin{aligned} F_{H1}^{(0)} &= -iku_0 u_1 - \alpha w_1\,\partial u_0/\partial Z, \\ F_{V1}^{(0)} &= -iku_0 w_1, \\ G_1^{(0)} &= -iku_0 r_1 + O(\alpha^5). \end{aligned}\right\} \tag{3.5}$$

Substituting these relations into (1.7), and using (2.13) and (2.14), we find that

$$M_1^{\pm} = \pm u_0 w_1 - \alpha ikw_1 \frac{\partial u_0}{\partial T} \mp \alpha iku_0 \frac{\partial w_1}{\partial Z} + O(\alpha^5). \tag{3.6}$$

Since u_0 is $O(\alpha^2)$, we have now confirmed that $M_1^{\pm}$ are $O(\alpha^3)$. Putting $m = 1$ in (2.3), it follows that the equation for w_1 is

$$L^{\pm}\left(\alpha\frac{\partial}{\partial T}, ik, \frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial Z}\right) w_1 = M_1^{\pm}. \tag{3.7}$$

Now u_0 is given by (2.28) and (2.33), and, at least to within an error $O(\alpha^5)$, u_0 is a function of (Z, T) only and is independent of z. The appropriate solution for w_1 is thus [cf. (2.7)]

$$w_1 = \alpha A^{\pm}(T, Z)\exp\{ik_c z\} + O(\alpha^5), \tag{3.8}$$

where

$$L^{\pm}\left(\alpha\frac{\partial}{\partial T}, ik, ik_c + \alpha\frac{\partial}{\partial Z}\right)A^{\pm} = \hat{M}_1^{\pm} + O(\alpha^4) \tag{3.9}$$

and

$$\hat{M}_1^{\pm} = \pm u_0 A^{\pm} - \alpha ik\frac{\partial u_0}{\partial T}A^{\pm} \mp \alpha iku_0\frac{\partial A^{\pm}}{\partial Z}. \tag{3.10}$$

Here $\hat{M}_1^{\pm}$ is $O(\alpha^2)$. The term $O(\alpha)$ in (3.9) is just (2.8), which in turn implies that u_0 is given by (2.39). In general, the equation (3.9) for $A^{\pm}$ and the equation (2.28) [with (2.33)] for u_0 are coupled. However, we may use (2.8), and then (3.9) to $O(\alpha^2)$, to approximate successively the higher derivatives of $A^{\pm}$ with respect to Z by derivatives with respect to T. Thus (3.9) may be recast in the form

$$\frac{\partial A^{\pm}}{\partial Z} = \mp 2\frac{\partial A^{\pm}}{\partial T} - 2\alpha ik_c\frac{\partial^2 A^{\pm}}{\partial T^2} \pm 4\alpha^2\frac{\partial^3 A^{\pm}}{\partial T^3} - \alpha ik_c\gamma A^{\pm} + \alpha N_1^{\pm} + O(\alpha^3), \tag{3.11}$$

where

$$\alpha^2 N_1^{\pm} = -2ik_c\hat{M}_1^{\pm} + \alpha\,\partial\hat{M}_1^{\pm}/\partial Z. \tag{3.12}$$

From (2.39) and (3.12) we have

$$N_1^{\pm} = 8ik_c|A^{\pm}|^2A^{\pm} + O(\alpha). \tag{3.13}$$

Next we may use (3.11) to $O(\alpha)$ in (2.33), and so express F_{H0} in terms of time derivatives. Then (2.28) implies, after some algebraic manipulation, that

$$u_0 = \mp 4\alpha^2|A^{\pm}|^2 + 12\alpha^3 ik\left(A^{\pm}\frac{\partial\bar{A}^{\pm}}{\partial T} - \bar{A}^{\pm}\frac{\partial A^{\pm}}{\partial T}\right) + O(\alpha^4). \tag{3.14}$$

This result may then be substituted into (3.10) and (3.12), and we find that

$$N_1^{\pm} = 8ik_c|A^{\pm}|^2A^{\pm} \pm 24\alpha(A^{\pm})^2\,\partial\bar{A}^{\pm}/\partial T + O(\alpha^2). \tag{3.15}$$

It is instructive to observe that (3.11) may be rewritten in the operational form

$$\alpha\,\partial A^{\pm}/\partial Z = (\pm 2i\omega + 2ik_c\omega^2 \pm 4i\omega^3 - ik_c\alpha^2\gamma)A^{\pm} + \alpha^2 N_1^{\pm} + O(\alpha^4), \tag{3.16}$$

where

$$\omega = i\alpha\,\partial/\partial T. \tag{3.17}$$

But, if we put $\omega = kc$ in $n^{\pm}$ in (1.10), and expand simultaneously about $\omega = 0$ and $k = k_c$ [or $\gamma = 0$, q.v. (1.25)], we can show that

$$\pm n^{\pm} = k_c - \alpha^2\gamma k_c \pm 2\omega + 2k_c\omega^2 \pm 4\omega^3 + O(\alpha^4). \tag{3.18}$$

Substituting (3.18) into (3.16) we obtain

$$\alpha\,\partial A^{\pm}/\partial Z = (\pm in^{\pm} - ik_c)A^{\pm} + \alpha^2 N_1^{\pm} + O(\alpha^4). \tag{3.19}$$

We have now shown that

$$Dw_1 = \frac{\partial w_1}{\partial z} + \alpha\frac{\partial w_1}{\partial Z} = (\pm in^{\pm}\alpha A^{\pm} + \alpha^3 N_1^{\pm})\exp\{ik_c z\} + O(\alpha^5). \tag{3.20}$$

Thus the result of totally differentiating w_1 with respect to z is the nonlinear term $N_1^{\pm}$ and the term obtained by applying the linear operator $\pm in^{\pm}$ to $A^{\pm}$; that the linear part should take this form is, of course, apparent from an examination of the unmodulated linear solution (1.14).

We are not in a position to consider the boundary conditions. First, however, we extract the Fourier component $m = 1$ from (1.2), and deduce that

$$ikp_1 = \mp iku_1 - \alpha\, \partial u_1/\partial T + F_{H1}. \tag{3.21}$$

If we now substitute for u_1 from (2.13) and use ω [see (3.17)] to represent the time derivative, we can show that

$$-k^2 p_1 = (\pm ik - i\omega)\left(\frac{\partial w_1}{\partial z} + \alpha \frac{\partial w_1}{\partial Z}\right) + ikF_{H1}. \tag{3.22}$$

Putting $m = 1$ in the boundary condition (2.5) and using (3.19), it follows that

$$-\alpha(k-\omega)\, n^+ A^+ + \alpha(k+\omega)\, n^- A^- = -k^2 Q_1 - ik[F_{H1}]^+_- - \alpha^3 i(k-\omega)\, N_1^+ - \alpha^3 i(k+\omega)\, N_1^- + O(\alpha^5). \tag{3.23}$$

Here $A^\pm$ etc. are evaluated at $z = 0\pm$. Next, putting $m = 1$ in the boundary condition (2.4), it follows that

$$\alpha A^\pm = \alpha(\pm ik - i\omega) A - H_1^\pm. \tag{3.24}$$

Substituting (3.23) into (3.22) then gives

$$\alpha \mathscr{D}(\omega, k) A = J, \tag{3.25}$$

where

$$\mathscr{D}(\omega, k) = -i(k-\omega)^2 n^+ - i(k+\omega)^2 n^-, \tag{3.26}$$

$$J = -k^2 Q_1 - ik[F_{H1}]^+_- - \alpha^3 i(k-\omega)\, N_1^+ - \alpha^3 i(k+\omega)\, N_1^- - (k-\omega)\, n^+ H_1^+ + (k+\omega)\, n^- H_1^- + O(\alpha^5). \tag{3.27}$$

Equation (3.25) is the amplitude equation we are seeking. Here J is a nonlinear expression in A, while $\mathscr{D}(\omega, k)$ is an ordinary linear differential operator with respect to T. If we ignore J, then (3.25) is formally equivalent to the dispersion relation (1.21) for the unmodulated linear case. From (3.18), we can show that

$$\mathscr{D} = -4i\omega(\omega^2 + \alpha^2\gamma) + O(\alpha^4). \tag{3.28}$$

Recalling the definition (3.17) of ω, it follows that

$$\alpha \mathscr{D}(\omega, k) A = 4\alpha^4 \frac{\partial}{\partial T}\left(-\frac{\partial^2 A}{\partial T^2} + \gamma A\right) + O(\alpha^5). \tag{3.29}$$

Equation (3.29) is the left-hand side of the amplitude equation (3.25), and a noteworthy feature is that it is $O(\alpha^4)$ and contains a third time derivative. This is a consequence of the fact that (1.21) (which is equivalent to formally equating $\mathscr{D}(\omega, k)$ to zero) has *three* solutions when k is near k_c. As (3.28) shows, one of these solutions has ω equal to zero and represents an internal gravity wave; the other two solutions are related to the instability which arises for k greater than k_c. Thus the unusual feature of an internal gravity wave co-existing with an instability is here reflected in the form of (3.29).

It remains to calculate the nonlinear term J. This is a long and tedious calculation and we shall only outline it here. Consider $H_1^\pm$; from (1.13) it follows that

$$H_1^\pm = -ik\{u_2 \bar{\zeta}_1 + \zeta_2 \bar{u}_1 + u_0 \zeta_1 + \tfrac{1}{2}\zeta_1^2 D\bar{u}_1 + |\zeta_1|^2 Du_1 + O(\alpha^5)\}. \tag{3.30}$$

Here u_1 and u_2 are evaluated at $z = 0\pm$. Then we use (2.7), (2.11), (2.13), (2.17), (2.24) and (2.25) to express (3.30) in terms of $A(T)$ alone. The result is

$$H_1^\pm = \pm\frac{7ik_c}{4}\alpha^3|A|^2A + \alpha^4(\mp 3ik_c - \tfrac{1}{2})|A|^2\frac{\partial A}{\partial T} + \tfrac{3}{4}\alpha^4A^2\frac{\partial \bar{A}}{\partial T} + O(\alpha^5). \quad (3.31)$$

This confirms that $H_1^\pm$ is $O(\alpha^3)$, a result which we anticipated earlier. From (1.12), Q_1 is given by

$$Q_1 = -\bar{\zeta}_1[Dp_2]_-^+ - \zeta_2[D\bar{p}_1]_-^+ - \tfrac{1}{2}\zeta_1^2[D^2\bar{p}_1]_-^+ - |\zeta_1|^2[D^2p_1]_-^+ + O(\alpha^5). \quad (3.32)$$

From (3.22), p_1 may be expressed in terms of w_1. Further, algebraic manipulation, using (2.7), (2.11), (2.17), (2.24) and (2.25), shows that

$$Q_1 = -8\alpha^4|A|^2\partial A/\partial T + O(\alpha^5). \quad (3.33)$$

Similarly, from (3.2) and (3.5) it may be shown that

$$ik_c[F_{H_1}]_-^+ = 6\alpha^4|A|^2\frac{\partial A}{\partial T} + 5\alpha^4A^2\frac{\partial \bar{A}}{\partial T} + O(\alpha^5). \quad (3.34)$$

Also, using (2.11), we can show that, at $Z = 0\pm$,

$$N_1^\pm = \mp 2|A|^2A + \alpha 8ik\left(|A|^2\frac{\partial A}{\partial T} + A^2\frac{\partial \bar{A}}{\partial T}\right) + O(\alpha^2). \quad (3.35)$$

We now substitute (3.31) and (3.33)–(3.35) into (3.27). Recalling that ω is $i\alpha\,\partial/\partial T$ [see (3.17)], we find that

$$J = 18\alpha^4|A|^2\frac{\partial A}{\partial T} + 8\alpha^4A^2\frac{\partial \bar{A}}{\partial T} + O(\alpha^5). \quad (3.36)$$

We have thus established that J is $O(\alpha^4)$, and since $\alpha\mathscr{D}A$ in (3.29) is also $O(\alpha^4)$, our choice of αT [see (1.26)] as the slow time variable has been justified.

4. Discussion of the amplitude equation

The amplitude equation is (3.25) with $\mathscr{D}$ given by (3.29) and the nonlinear term J given by (3.36). Hence the amplitude equation is

$$-\frac{\partial^3 A}{\partial T^3} + \gamma\frac{\partial A}{\partial T} = \tfrac{9}{2}|A|^2\frac{\partial A}{\partial T} + 2A^2\frac{\partial \bar{A}}{\partial T}. \quad (4.1)$$

As the right-hand side of this equation is not integrable, we have not been able to obtain its general solution in an explicit form. However, by inspection we see that (4.1) has the particular solution

$$A = A_0, \quad (4.2)$$

where A_0 is an *arbitrary* (complex) constant. The particular solution (4.2) may be interpreted as a finite amplitude internal gravity wave, which is stationary with respect to the interface.

The stability of this solution may be determined by putting

$$A = A_0(1 + B), \quad (4.3)$$

and substituting this into (4.1), so that

$$-\frac{\partial^3 B}{\partial T^3}+\gamma\frac{\partial B}{\partial T} = |A_0|^2\left\{\tfrac{9}{2}|1+B|^2\frac{\partial B}{\partial T}+2(1+B)^2\frac{\partial \bar{B}}{\partial T}\right\}. \tag{4.4}$$

The equations which determine the stability of A_0 are obtained by linearizing this equation with respect to B:

$$\frac{\partial}{\partial T}\left\{-\frac{\partial^2 B_R}{\partial T^2}+\gamma B_R-\tfrac{13}{2}|A_0|^2 B_R\right\} = 0, \tag{4.5}$$

$$\frac{\partial}{\partial T}\left\{-\frac{\partial^2 B_I}{\partial T^2}+\gamma B_I-\tfrac{5}{2}|A_0|^2 B_I\right\} = 0, \tag{4.6}$$

where

$$B = B_R + iB_I. \tag{4.7}$$

Thus the particular solution (4.2) is stable to small perturbations if

$$|A_0|^2 > \tfrac{2}{5}\gamma. \tag{4.8}$$

This condition will always be satisfied if γ is negative; this is to be expected as then k is less than k_c [see (1.25)] and the interface $z = 0$ is stable according to linearized theory. When γ is positive the interface $z = 0$ is unstable according to linearized theory. Nevertheless (4.8) shows that the interface can support a stable finite amplitude internal gravity wave; the amplitude of this wave, $\alpha|A_0|$, must satisfy (4.8), or from (1.25),

$$\tfrac{5}{2}|\alpha A_0|^2 > (k-k_c)/k_c. \tag{4.9}$$

Lindzen (1974) discussed four observations of internal gravity waves associated with shear zones, for which the present model may be applicable. The same four observations are reproduced in table 1, and the observed values of $\alpha|A_0|$ are compared with the observed values of $(k-k_c)/k_c$. Note that the crest-to-trough amplitude is $4\alpha|A_0|$. Also it is difficult to determine a precise value of U and N from the observations, as the observed shear layers are seldom simple discontinuities; consequently the values of U and N quoted should be regarded as no more than representative. In the first two cases, k is less than k_c and so (4.9) is trivially satisfied; in the remaining two cases k is greater than k_c and (4.9) is again satisfied. However, other observations, not reproduced here, show that much shorter wavelengths are also observed and then (4.9) is not satisfied; presumably these other observations are cases for which the length scale associated with the width of the shear zone (ignored in the present model) is more important than the length scale UN^{-1}. The equation of the interface associated with the particular solution (4.2) may be determined from (2.10) and (2.24), and is given by

$$\zeta = 2\alpha|A_0|\cos(kx+\phi_0)-2\alpha^2|A_0|^2\sin(2kx+2\phi_0)+O(\alpha^3), \tag{4.10}$$

where

$$\phi_0 = \arg A_0.$$

Figure 2 shows the graph of ζ compared with the sinusoidal graph obtained from linear theory. The graph shows that the effect of the nonlinear terms is to increase the slope on one face of the wave and to decrease it on the other face. This is

Case	U (m s^{-1})	N (s^{-1})	UN^{-1} (m)	Observed wave-length (km)	Observed wave-length (units of UN^{-1})	Observed wave-number k (units of UN^{-1})	$\frac{k-k_c}{k_c}$	Observed amplitude (m)	Observed amplitude $\alpha\|A_0\|$ (units of UN^{-1})	$\frac{5}{2}\|\alpha A_0\|^2$
Wallops Island, Va. 7 February 1968 (Ottersten *et al.* 1973)	10	2×10^{-2}	500	6	12	0·52	−0·26	125	0·25	0·16
Haswell, Colo. 12 November 1971 (Hooke, Hall & Gossard 1973)	1	$2{\cdot}7\times10^{-2}$	37	0·35	9·45	0·66	−0·07	7·5	0·20	0·10
Wallops Island, Va. 19 February 1970 (Hardy, Reed & Mather 1973)	7	2×10^{-2}	350	2·7	7·7	0·81	0·15	125	0·36	0·32
Wallops Island, Va. 18 March 1969 (Reed & Hardy 1972)	24	$1{\cdot}3\times10^{-2}$	1850	15–20	8·1–10·8	0·77–0·58	0·09–−0·18	450	0·24	0·15

TABLE 1. Observed amplitudes and wavelengths of internal gravity waves associated with shear zones

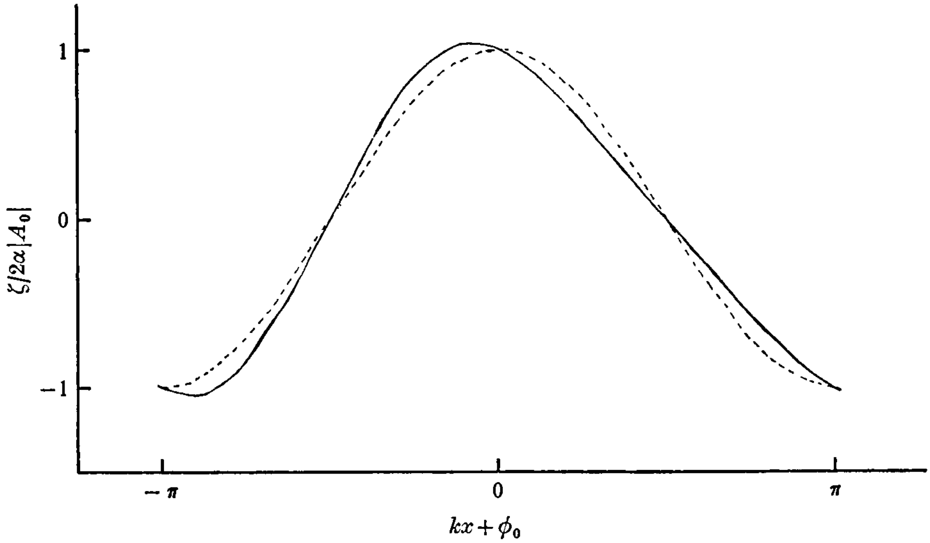

FIGURE 2. The graph of ζ determined by (4.10) when $\alpha|A_0|$ is 0·2 (solid line): the corresponding sinusoidal graph is shown dashed.

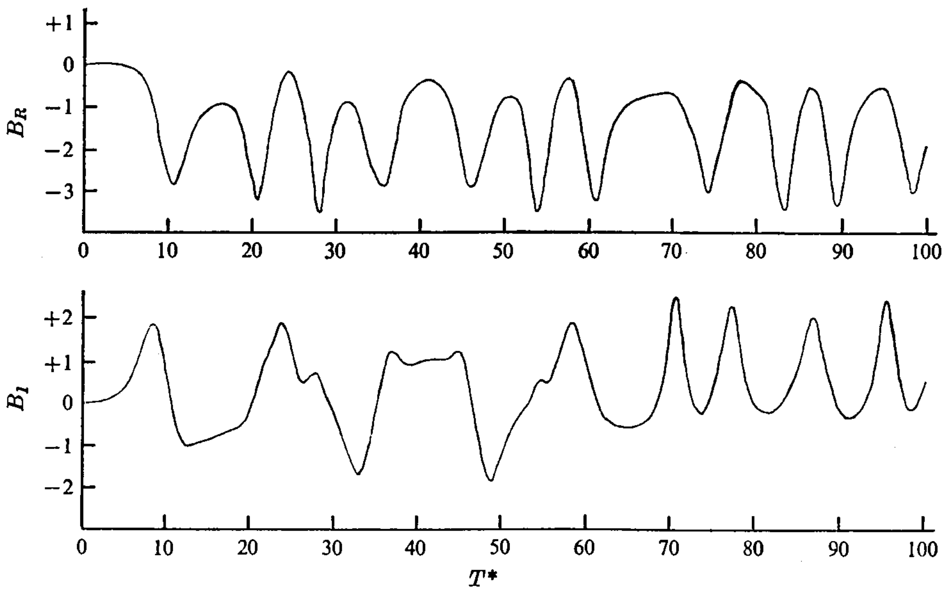

FIGURE 3. The computed evolution of B with time T^*. The graphs displayed are for $|A_0|^2/\gamma = 0{\cdot}2$ and for the following initial conditions at $T^* = 0$: $B = 0{\cdot}01(1+2i)$, $\partial B/\partial T^* = 0{\cdot}0055$, $\partial^2 B/\partial T^{*2} = 0{\cdot}01(-0{\cdot}3+i)$.

a characteristic feature of many observations of internal gravity waves (q.v. review by Ottersten *et al.* 1973).

When γ is positive, we put

$$T^* = \gamma^{\frac{1}{2}} T \tag{4.11}$$

in (4.4), which then becomes

$$-\frac{\partial^3 B}{\partial T^{*3}} + \frac{\partial B}{\partial T^*} = \frac{|A_0|^2}{\gamma}\left\{\tfrac{9}{2}|1+B|^2 \frac{\partial B}{\partial T^*} + 2(1+B)^2 \frac{\partial \bar{B}}{\partial T^*}\right\}. \tag{4.12}$$

This equation was integrated numerically with initial conditions such that

$$|B|, \left|\frac{\partial B}{\partial T^*}\right|, \left|\frac{\partial^2 B}{\partial T^{*2}}\right| \ll 1, \quad T^* = 0. \tag{4.13}$$

These initial conditions represent a small perturbation of the solution (4.2). When the stability criterion (4.8) is satisfied, the solution for B is a small amplitude oscillation, and is well described by (4.5) and (4.6). However, when (4.8) is not satisfied, the solution for B soon evolves into a large amplitude, irregular oscillation. These oscillations depend on the parameter $|A_0|^2/\gamma$, and also show some sensitivity to the initial conditions. A typical oscillation is shown in figure 3. It is a natural conjecture that the solutions are oscillating about a constant equilibrium B_1, where B_1 is stable and so satisfies the condition

$$|A_0|^2 |1+B_1|^2 > \tfrac{2}{5}\gamma. \tag{4.14}$$

In order to test this conjecture, a damping term

$$-\kappa \partial^2 B/\partial T^{*2} \tag{4.15}$$

was inserted into the left-hand side of (4.12), where κ, the damping coefficient, is small and positive. The numerical solutions then converged to a value B_1, which was observed to satisfy (4.14). The results are displayed in table 2. The constant B_1 was found to depend solely on $|A_0|^2/\gamma$, and was independent both of the damping coefficient (always less than 0·1) and of the initial conditions (4.13). We have been unable to determine any analytic relation between B_1 and $|A_0|^2/\gamma$, although the computed values have the intriguing feature that $\arg(1+B_1)$ is approximately $\frac{3}{4}\pi$ in every case. It is also apparent from table 2 that B_1 is such that (4.14) is only narrowly satisfied.

Finally, we exhibit some other particular solutions of (4.1), or equivalently (4.4). First, it may easily be verified that (4.1) has a periodic solution

$$A = A_0 \exp(i\mu T), \tag{4.16}$$

where $\mu^2 + \gamma = \frac{5}{2}|A_0|^2.$

Here μ must be real, and so the amplitude A_0 must satisfy (4.8). It can be shown that this periodic solution is stable to small perturbations. However, none of the computed solutions showed any tendency to converge to (4.16). Second, if the initial conditions are such that B_I, the imaginary part of B, is always zero, then (4.4) may be integrated. Note, however, from (4.5) and (4.6), that in the unstable case perturbations in B_I grow more rapidly than perturbations in B_R, the real part of B; indeed, it is the perturbations in B_I which determine the stability

$\lvert A_0\rvert^2/\gamma$	B_1	$\frac{5}{2}\lvert A_0\rvert^2\lvert 1+B_1\rvert^2/\gamma$
0·36	$-1{\cdot}80+i0{\cdot}70$	1·02
0·32	$-1{\cdot}90+i0{\cdot}73$	1·07
0·28	$-1{\cdot}99+i0{\cdot}78$	1·11
0·24	$-2{\cdot}12+i0{\cdot}88$	1·22
0·20	$-2{\cdot}26+i0{\cdot}98$	1·27
0·16	$-2{\cdot}46+i1{\cdot}04$	1·29
0·13	$-2{\cdot}62+i1{\cdot}16$	1·29
0·10	$-2{\cdot}77+i1{\cdot}37$	1·25
0·07	$-2{\cdot}98+i1{\cdot}70$	1·19
0·04	$-3{\cdot}50+i2{\cdot}30$	1·15

TABLE 2. The computed equilibrium values B_1. The damping coefficient κ was set at 0·05 for the first five cases, and at 0·075 for the last five cases

criterion. Nevertheless it is of some interest to examine the solution when B_I is identically zero. Then (4.4) becomes

$$-\frac{\partial^3 B_R}{\partial T^3}+\gamma\frac{\partial B_R}{\partial T} = |A_0|^2\tfrac{13}{2}(1+B_R)^2\frac{\partial B_R}{\partial T}. \tag{4.17}$$

This equation may be integrated once, and hence

$$-\frac{\partial^2 B_R}{\partial T^2}+\gamma B_R = |A_0|^2\tfrac{13}{6}B_R(B_R^2+3B_R+3). \tag{4.18}$$

Here we have put the constant of integration equal to zero; it can be shown that this may always be achieved by translating the origin for B_R and adjusting the value of $|A_0|$. Equation (4.18) can be analysed by standard phase-plane techniques. The equliibrium points are

$$B_R = 0, \quad B_R = B_R^{\pm} = -\tfrac{3}{2}\pm\left(\frac{6\gamma}{13|A_0|^2}-\tfrac{3}{4}\right)^{\frac{1}{2}}$$

if

$$|A_0|^2 < \tfrac{8}{13}\gamma, \tag{4.19}$$

and just $B_R = 0$ if $|A_0|^2$ is greater than $\frac{8}{13}\gamma$. If $|A_0|^2 < \frac{2}{13}\gamma$, the equilibrium point $B_R = 0$ is unstable, while $B_R^{\pm}$ are both stable; small perturbations in B_R evolve into large amplitude oscillations about $B_R^{\pm}$. If $|A_0|^2 > \frac{2}{13}\gamma$, the equilibrium point $B_R = 0$ is stable, while B_R^{+} is unstable and B_R^{-} is stable; small perturbations in B_R evolve into small amplitude oscillations about zero.

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